

MATH 579 Exam 9 Solutions

1. Use generating functions to solve the recurrence given by $a_0 = 3, a_n = 3a_{n-1} - 4 (n \geq 1)$.

$\sum_{n \geq 1} a_n x^n = \sum_{n \geq 1} 3a_{n-1} x^n - 4 \sum_{n \geq 1} x^n$, hence $A(x) - 3 = 3x \sum_{n \geq 1} a_{n-1} x^{n-1} - 4x \sum_{n \geq 1} x^{n-1} = 3xA(x) - \frac{4x}{1-x}$. A bit of algebra, then partial fractions, gives $A(x) = \frac{3-7x}{(1-x)(1-3x)} = \frac{2}{1-x} + \frac{1}{1-3x}$. Hence $A(x) = \sum_{n \geq 0} 2x^n + \sum_{n \geq 0} 3^n x^n = \sum_{n \geq 0} (2 + 3^n)x^n$, so $a_n = 2 + 3^n$.

2. Use generating functions to solve the recurrence given by $a_0 = 1, a_1 = 2, a_n = 2a_{n-1} - a_{n-2} (n \geq 2)$.

$\sum_{n \geq 2} a_n x^n = \sum_{n \geq 2} 2a_{n-1} x^n - \sum_{n \geq 2} a_{n-2} x^n$, hence $A(x) - 2x - 1 = 2x \sum_{n \geq 2} a_{n-1} x^{n-1} - x^2 \sum_{n \geq 2} a_{n-2} x^{n-2} = 2x(A(x) - 1) - x^2 A(x)$. A bit of algebra gives $A(x) = \frac{1}{(1-x)^2}$. Hence $A(x) = \sum_{n \geq 0} (n+1)x^n$, so $a_n = n+1$.

3. $A(x) = \frac{x^4 + 5x - 2}{(1-2x)^5}$ is the generating function for a sequence a_n . Find a closed form for a_n (for $n \geq 4$ is sufficient).

$A(x) = (x^4 + 5x - 2) \sum_{n \geq 0} \binom{n+4}{4} 2^n x^n = \sum_{n \geq 0} \binom{n+4}{4} 2^n x^{n+4} + 5 \sum_{n \geq 0} \binom{n+4}{4} 2^n x^{n+1} - 2 \sum_{n \geq 0} \binom{n+4}{4} 2^n x^n = \sum_{n \geq 4} \binom{n}{4} 2^{n-4} x^n + 5 \sum_{n \geq 1} \binom{n+3}{4} 2^{n-1} x^n - 2 \sum_{n \geq 0} \binom{n+4}{4} 2^n x^n = \sum_{n \geq 4} \binom{n}{4} 2^{n-4} x^n + 5 \sum_{n \geq 4} \binom{n+3}{4} 2^{n-1} x^n - 2 \sum_{n \geq 4} \binom{n+4}{4} 2^n x^n + p(x)$, where $p(x)$ is some polynomial of degree at most 3, which won't affect the terms we care about (with $n \geq 4$). Continuing, we get: $p(x) + \sum_{n \geq 4} ((\binom{n}{4} 2^{-4} + 5 \binom{n+3}{4} 2^{-1} - 2 \binom{n+4}{4})) 2^n x^n$. Hence, for $n \geq 4$, $a_n = (\frac{1}{16} \binom{n}{4} + \frac{5}{2} \binom{n+3}{4} - 2 \binom{n+4}{4}) 2^n$.

4. Find a generating function $V(x)$ that can be used to count nonnegative integer solutions to $a+b+c = n$, where (1) $2 \leq a \leq 5$, (2) b is a multiple of 5, (3) c is odd. You should simplify $V(x)$, but DO NOT ATTEMPT TO FIND A CLOSED FORM FOR THE SEQUENCE.

This will be a product of three generating functions, one for each piece. $A(x) = x^2 + x^3 + x^4 + x^5 = x^2(1+x)(1+x^2)$. $B(x) = \frac{1}{1-x^5}$. If c were even, we would want $\frac{1}{1-x^2}$ but instead we want it shifted by one, so $C(x) = \frac{x}{1-x^2}$. Hence $V(x) = A(x)B(x)C(x) = \frac{x^3(1+x)(1+x^2)}{(1-x^5)(1-x^2)}$. Stopping here is wise, since the partial fractions are very messy.

5. Vadim's Amazing Magic Trick begins with a deck of n cards. Someone from the audience is chosen to be the assistant. This assistant separates the deck into any number of nonempty piles. From each pile the assistant chooses one card to memorize. Also, the assistant places the Magic Vodka Bottle on top of one of the piles. Let v_n denote the number of ways the assistant can do all this. $v_0 = 1, v_1 = 1, v_2 = 4$ (if two piles, two choices for MVB; if one pile, two choices for card to memorize). Find the generating function $V(x)$. DO NOT ATTEMPT TO FIND A CLOSED FORM FOR THE SEQUENCE.

This is an application of the Composition Theorem, 8.15. We divide the interval $[n]$ into subintervals. On each we do operation A (pick a card), and then on the subintervals we do operation B (pick a subinterval). The answer will be $V(x) = B(A(x))$. We have $A(x) = \sum_{n \geq 0} nx^n = \sum_{n \geq 1} nx^n = x \sum_{n \geq 1} nx^{n-1} = x \sum_{n \geq 0} (n+1)x^n = \frac{x}{(1-x)^2}$. Note that $a_0 = 0$, as required for the Composition Theorem. By the Composition Theorem, we must have $b_0 = 1$. This means that if $n = 0$, there are no cards at all, and in this special case we're not obligated to place the MVB (we can just drink it, since we don't have any cards and we are sad). Hence $B(x) = 1 + \sum_{n \geq 0} nx^n = 1 + \frac{x}{(1-x)^2}$. $V(x) = B(A(x)) = 1 + \frac{\frac{x}{(1-x)^2}}{(1-\frac{x}{(1-x)^2})^2}$.

If you were ambitious, you could simplify $V(x) = \frac{x^4 - 5x^3 + 9x^2 - 5x + 1}{(x^2 - 3x + 1)^2}$. Doing partial fractions on this by hand is not a good use of time.